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14. ABSTRACT

A chirp transform is defined for the Δ-step LMS adaptive predictors for linearly chirped signals embedded in additive white Gaussian noise. By converting the chirped signals to stationary baseband signals, this transform provides a different approach in analyzing the tracking performance of the LMS adaptive predictors. This transform also provides an approach of analyzing the nonlinear effects of the LMS adaptive predictor for nonstationary input signals. It is also shown that the chirp transform can be applied to the 1-step RLS predictor with chirped input signals.

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Chirp Transform in the Nonlinear Tracking Performance Analysis of the LMS Adaptive Predictors

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Abstract[†]

A chirp transform is defined for the Δ-step LMS adaptive predictors for linearly chirped signals embedded in additive white Gaussian noise. By converting the chirped signals to stationary baseband signals, this transform provides a different approach in analyzing the tracking performance of the LMS adaptive predictors. This transform also provides an approach of analyzing the nonlinear effects of the LMS adaptive predictor for nonstationary input signals. It is also shown that the chirp transform can be applied to the 1-step RLS predictor with chirped input signals.

1. Introduction

The tracking behavior of adaptive filtering algorithms is a fundamental issue in defining their performance in nonstationary operating environments. The convergence behavior of an adaptive filter is a transient phenomenon whereas the tracking behavior is a steady-state property[1]. [2] studied one-step Least-Mean-Square (LMS) adaptive predictor when the input signal consists of a chirped sinusoid in Additive White Gaussian Noise (AWGN). In [3] the performance of the one-step LMS predictor is analyzed when the input is a chirped first order Autoregressive process (AR1) embedded in AWGN.

Traditional analysis of the LMS adaptive filter performance, including the analysis in the above references, is restricted to a statistical analysis of the algorithm under a set of independence assumptions that ignore the statistical dependence among successive tap-input vectors [4]. The Mean-Squared Error (MSE) of the

LMS adaptive filter using these assumptions is bounded by that of the corresponding finite-length Wiener filter. These well-known assumptions mask the nonlinear effects that arise in LMS adaptive filters. It has been shown that it is possible for the LMS adaptive filter to outperform the finite-length Wiener filter in MSE for the cases of adaptive noise cancellation [5], adaptive channel equalization [6], and adaptive prediction [7][8]. An error transfer function approach is also derived in [6] to give an approximate expression for the total steady-state MSE of the LMS adaptive channel equalizer. A chirp transform is defined in [7][8] to convert the chirped AR1 signal to stationary signal and approximate the steady-state MSE of the LMS algorithms in the transformed domain.

In this paper, we show that the chirp transform defined in [7][8] can be applied to multiple-step LMS predictors with linearly chirped signal, where the original signal can be an arbitrary signal, not necessarily AR1 process. The chirp transform will convert the linearly chirped signal to its original form, and the analysis of steady-state MSE can be done on stationary signals. The chirp transform can also be used to the exponentially weighted Recursive-Least-Square (RLS) algorithm.

2. Background

The adaptive predictor application considered is the adaptive recovery of narrowband signals from embedded AWGN. Fig. 1 represents the linear Δ-step adaptive predictor structure to be analyzed, where $\mathbf{w}^c(n) = [w_1^c \quad w_2^c \quad \cdots \quad w_M^c]^T$ are the adaptive filter weights; x_n^c is the input to the adaptive predictor, $x_n^c = s_n^c + n_n, n = 0, 1, 2, \ldots$; and s_n^c has initial center frequency ω_0 and chirp rate ψ. s_n^c is the linearly chirped signal from the stationary baseband signal s_n,

$$s_n^c = \Omega^n \Psi^{\frac{n^2}{2}} s_n, \quad n = 0, 1, 2, \ldots, \quad \text{where} \quad \Omega = e^{j\omega_0}, \quad \text{and}$$
$$\Psi = e^{j\psi}.$$

The weight update equation of the LMS algorithm is

$$\mathbf{w}^c(n+1) = \mathbf{w}^c(n) + \mu \mathbf{x}^{c*}(n) e_n^c, \qquad (1)$$

where μ is the step-size parameter of the adaptive algorithm, $\mathbf{x}^c(n)$ is the adaptive filter input tap-vector at time index n, and * denote the complex conjugate. For the Δ-step predictor,

$$\mathbf{x}^c(n) = \begin{bmatrix} x_{n-\Delta}^c \\ x_{n-(\Delta+1)}^c \\ \vdots \\ x_{n-(\Delta+M-1)}^c \end{bmatrix}. \qquad (2)$$

The error update equation is given by

$$e_{n+1}^c = x_{n+1}^c - \mathbf{w}^{cT}(n+1)\mathbf{x}^c(n+1). \qquad (3)$$

3. Chirp Transform for LMS Predictor

The error transfer function approach derived in [6] provides a method to approximate the total steady-state MSE of the LMS adaptive filter without explicitly invoking the independence assumptions for wide-sense stationary input signals, i.e., signals with a fixed PSD. For a chirped input signal x_n^c, the PSD is constantly shifting with time, and this approach is not directly applicable. However, by multiplying the chirped input signal by a negative frequency offset sequence, we can transform the chirped signal s_n^c to its stationary baseband form s_n and leave the noise component n_n unchanged in statistics since AWGN has a constant spectral envelope across all frequencies. In the following, it is shown that the above transform will not change the MSE of the LMS adaptive predictor for a chirped input signal. This allows the error transfer function approach to be applied to rotated LMS algorithm with the transformed input signals in order to approximate the MSE of the standard LMS adaptive predictor with chirped input signals.

3.1 Equivalence of MSEs

Multiplying (3) by $\Omega^{-(n+1)}\Psi^{-\frac{(n+1)^2}{2}}$, and defining $e_n'' = \Omega^{-n}\Psi^{-\frac{n^2}{2}} e_n^c$, $n = 0, 1, 2, \ldots$ (3) becomes

$$e_{n+1}'' = x_{n+1}'' - \mathbf{w}''^T(n+1)\mathbf{x}''(n+1). \qquad (4)$$

where $\mathbf{w}''(n+1)$ is the corresponding predictor weight in the transformed domain and $\mathbf{x}''(n+1)$ is corresponding unchirped stationary input signal vector. Note that for the unchirped signal $x_n'' = s_n + n_n''$, $n_n'' = \Omega^{-n}\Psi^{-\frac{n^2}{2}} n_n$, $n = 0, 1, 2, \ldots n_n''$ and n_n are AWGN with the same statistics. (1) becomes

$$\mathbf{w}''(n+1) = \mathbf{V}_\Lambda^* [\mathbf{w}''(n) + \mu \mathbf{x}''^*(n) e_n''], \qquad (5)$$

where

$$\mathbf{V}_\Lambda \triangleq \Psi^{\Lambda-1} * \text{diag}\{\Psi^1, \Psi^2, \cdots, \Psi^M\} \qquad (6)$$

is the chirp rotation matrix. Since e_n'' is the transformed version of e_n^c, they have the same power, i.e.,

$$E[|e_n''|^2] = E[|\Omega^{-n}\Psi^{-\frac{n^2}{2}} e_n^c|^2] = E[|e_n^c|^2]. \qquad (7)$$

Consequently, the MSE of the LMS adaptive predictor with a chirped input signal x_n^c is equal to the MSE of a different LMS adaptive predictor with a corresponding stationary baseband input signal x_n''. The two adaptive predictors have the same length M and step-size μ. Equations (4) and (5) define the error and weight vector updates of the rotated LMS adaptive predictor. The only difference between these equations and the standard LMS adaptive predictor for stationary input signals as in (1) and (3) is that the weight vector is rotated in frequency by the chirp matrix $\mathbf{V}_\Lambda^*$ after each normal LMS update, as shown in Fig. 2.

3.2 Error Transfer Function Approach for the Rotated LMS Adaptive Predictor

Decompose the rotated LMS adaptive predictor weight into the sum of a time-invariant finite-length Wiener predictor weight and a time-varying misadjustment component

$$\mathbf{w}''(n) = \mathbf{w}_0 + \mathbf{w}_{mis}''(n). \qquad (8)$$

$\mathbf{w}_{mis}''(n)$ is further decomposed as

$$\mathbf{w}_{mis}''(n) = \bar{\mathbf{w}}_{mis}''(n) + \tilde{\mathbf{w}}_{mis}''(n), \qquad (9)$$

where $\bar{\mathbf{w}}_{mis}''(n) = E[\mathbf{w}_{mis}''(n)]$ is the mean weight misadjustment corresponding to the weight fluctuation caused by weight rotation. From (5), the mean weight misadjustment is given by

$$\overline{\mathbf{w}}_{mis}^{n}(n+1) = \mathbf{V}_{\Lambda}^{*}(\mathbf{I} - \mu\mathbf{R})\overline{\mathbf{w}}_{mis}^{n}(n) - (\mathbf{I} - \mathbf{V}_{\Lambda}^{*})\mathbf{w}_{0}, \quad (10)$$

where $\mathbf{R}$ is the autocorrelation matrix of the stationary input signal. When $n \to \infty$, i.e., the adaptive filter reaches steady state,

$$\overline{\mathbf{w}}_{mis}^{n} = \overline{\mathbf{w}}_{mis}^{n}(\infty) = -(\Lambda + \mu\mathbf{R})^{-1}\Lambda\mathbf{w}_{0}, \quad (11)$$

where $\Lambda \triangleq \mathbf{V}_{\Lambda} - \mathbf{I}$. Note that in (10), it is assumed $\overline{\mathbf{w}}_{mis}^{n}(n)$ to be independent of $\mathbf{x}^{n*}(n)\mathbf{x}^{nT}(n)$, and it is not necessary for $\overline{\mathbf{w}}_{mis}^{n}(n)$ to be independent of $\mathbf{x}^{n}(n)$. This steady-state mean weight misadjustment term corresponds to the lag weight misadjustment of the LMS adaptive predictor with a chirped input process as shown in [3].

The weight update equation (5) can be written as

$$\mathbf{w}^{n}(n) = \mathbf{V}_{\Lambda}^{*n}\mathbf{w}^{n}(0) + \mu\sum_{j=0}^{n-1}\mathbf{V}_{\Lambda}^{*(n-j)}\mathbf{x}^{n*}(j)e_{j}^{n}. \quad (12)$$

At steady state, $\mathbf{V}_{\Lambda}^{*n}\mathbf{w}^{n}(0)$ can be replaced with $\mathbf{w}_{0} + \overline{\mathbf{w}}_{mis}^{n}$, thus the error process e_{n}^{n} satisfies the recursive difference equation

$$e_{n}^{n} + \mu\sum_{j=0}^{n-1}e_{j}^{n}\mathbf{x}^{nH}(j)\mathbf{V}_{\Lambda}^{*(n-j)}\mathbf{x}^{n}(n)$$
$$= x_{n}^{n} - [\mathbf{w}_{0} + \overline{\mathbf{w}}_{mis}^{n}]^{T}\mathbf{x}^{n}(n) \quad (13)$$

Using the approximation [6]

$$\mathbf{x}^{nH}(j)\mathbf{x}^{n}(n) \approx Mr_{x}^{n}(n-j), \quad (14)$$

and

$$V_{\Lambda} \approx \psi^{\Delta + \frac{M-1}{2}}\mathbf{I}, \qquad \psi \ll 1, \quad (15)$$

where $r_{x}^{n}(k)$ is the autocorrelation of the stationary input signal x_{n}^{n}, we have,

$$\mathbf{x}^{nH}(j)\mathbf{V}_{\Lambda}^{*(n-j)}\mathbf{x}^{n}(n) \approx Mr_{x}^{c}(n-j), \quad (16)$$

where

$$r_{x}^{c}(n-j) \triangleq \psi^{-(\Lambda + \frac{M-1}{2})(n-j)}r_{x}^{n}(n-j). \quad (17)$$

Equation (13) can thus be approximated by a standard difference equation with constant coefficients as

$$e_{n}^{n} + \mu M\sum_{j=0}^{n-1}r_{x}^{c}(n-j)e_{j}^{n} = x_{n}^{n} - [\mathbf{w}_{0} + \overline{\mathbf{w}}_{mis}^{n}]^{T}\mathbf{x}^{n}(n). \quad (18)$$

We can interpret the steady-state ($n \to \infty$) rotated LMS adaptive predictor error e_{n}^{n} as the output of a time-invariant linear system with transfer function $H(z)$

driven by the wide-sense stationary error process $x_{n}^{n} - [\mathbf{w}_{0} + \overline{\mathbf{w}}_{mis}]^{T}\mathbf{x}^{n}(n)$, where $H(z)$ is given by

$$H(z) = \frac{1}{1 + \mu MR(z)}, \quad (19)$$

where

$$R(z) = \sum_{m=1}^{\infty}r_{x}^{c}(m)z^{-m} \quad (20)$$

The steady-state MSE of the rotated LMS adaptive predictor is thus

$$J_{lms} = \frac{1}{2\pi j}\oint_{|z|=1}|H(z)|^{2}\left|1 - \mathbf{w}_{0}(z) - \overline{\mathbf{w}}_{mis}^{n}(z)\right|^{2}S_{xx}^{n}(z)\frac{dz}{z},$$

where $W_{0}(z) = \sum_{j=\Delta}^{M+\Delta-1}w_{0,j}z^{-j}$, $\overline{W}_{mis}^{n}(z) = \sum_{j=\Delta}^{M+\Delta-1}\overline{w}_{mis,j}^{n}z^{-j}$ are the transfer functions of the finite-length Wiener predictor and mean weight misadjustment of the rotated LMS adaptive predictor respectively. $S_{xx}^{n}(z)$ is the PSD of the stationary input process x_{n}^{n}.

The error transfer function approach can also be applied to the Normalized LMS (NLMS) algorithm as defined below [6]

$$\mathbf{w}^{c}(n+1) = \mathbf{w}^{c}(n) + \frac{\mu}{\|\mathbf{x}^{c}(n)\|^{2}}\mathbf{x}^{c*}(n)e_{n}^{c} \quad (22)$$

with

$$H(z) = \frac{1}{1 + \mu R(z)/(P_{x} + P_{n})}. \quad (23)$$

4. Chirp Transform for RLS Predictor

In this section, only the 1-step RLS predictor is studied due to the difficulty involved in the inversion of the autocorrelation matrix estimate. The weight and error update equations of the exponentially weighted RLS adaptive algorithm is given by [4] and (3)

$$\mathbf{w}^{c}(n+1) = \mathbf{w}^{c}(n) + [\Phi_{x}^{c}(n)]^{-1}\mathbf{x}^{c*}(n)e_{n}^{c}, \quad (24)$$

where $\Phi^{c}(n) = \sum_{i=0}^{n}\lambda^{n-i}\mathbf{x}^{c*}(n)\mathbf{x}^{cT}(n)$ is the input signal autocorrelation matrix estimate at time n, and λ is the forgetting factor of the RLS algorithm.

In order to relate the inverse of autocorrelation estimations $[\Phi_{x}^{c}(n)]^{-1}$ of chirped process in (24) to the autocorrelation estimate $[\Phi_{n}(n)]^{-1}$ of the baseband

stationary process $x_n'' = s_n + n_n''$, use steady-state approximation $[\Phi_x^{-1}(n)]^{-1} \approx (1-\lambda)\mathbf{R}^{-1}$, and define a signal direction matrix

$$\mathbf{D} = \text{diag}\{\Omega^{\Psi^{-\frac{1}{2}}}, \Omega^2\Psi^{-\frac{2^2}{2}}, ..., \Omega^M\Psi^{-\frac{M^2}{2}}\} \quad (25)$$

Then at steady-state,

$$[\Phi_x^c(n)]^{-1} \approx \mathbf{V}^{n+1}\mathbf{D} \ [\Phi_x(n)]^{-1}\mathbf{D}^H\mathbf{V}^{*(n+1)} \quad (26)$$

where $\mathbf{V} = \mathbf{V}_\Lambda \mid_{\Delta=1}$. (24) becomes

$$\mathbf{w}^c(n+1) = \mathbf{w}^c(n) + \mathbf{V}^{n+1}\mathbf{D}[\Phi_x(n)]^{-1}\mathbf{D}^*\mathbf{V}^{*(n+1)}\mathbf{x}^{*c}(n)e_n^c$$

It can be simplified to

$$\mathbf{w}''(n+1) = \mathbf{V}^*[\mathbf{w}''(n) + \mathbf{V} \ [\Phi_x(n)]^{-1}\mathbf{V}^*\mathbf{x}''^*(n)e_n'']\quad (27)$$

The MSE of RLS on chirped process x_n^c can be approximated by the MSE of RLS on a correspondence stationary baseband process x_n with an adaptive filter of the same length M, forgetting factor λ. The differences of equations (4) and (27) with the standard RLS algorithm for stationary signal (3) and (24) is that the weight vector is rotated by chirp matrix $\mathbf{V}^*$ after each update and the inverse of autocorrelation matrix estimation is pre-rotated by $\mathbf{V}$ and post-rotated by $\mathbf{V}^*$ in each update. The rotations will give extra MSE in excess of the normal MSE of standard RLS algorithms.

5. Simulations

Several simulations have been performed to show the equivalence of MSEs using chirp transform for Δ-step LMS predictor and 1-step RLS predictor using the chirped AR1 signals, and the transfer function approach in approximating the MSE of the LMS predictor with chirped AR1 signals.

Fig. 3 is a plot of MSEs of Wiener predictor, standard LMS with chirped AR1 input signal, and rotated LMS with corresponding stationary input signal. The signal is pole location is at $a = 0.9$, input signal SNR=5dB, filter length $M = 10$, the prediction distance is 5, the chirp rate is $\psi = 5e-4$. It shows from the Fig. that the MSEs are essentially the same for standard LMS on chirped input signal and rotated LMS on stationary input signal.

Fig. 4 plots MSEs of 40-step NLMS predictors as a function of adaptation constant, with SNR = 20dB, M=25, a = 0.99, chirp rate $\psi = 5\pi e-4$. In this plot, the NLMS adaptive predictors are used instead of the standard LMS adaptive predictors because the NLMS algorithm is stable for relatively larger values of the adaptive filter step-size ($0 < \mu < 2$) [6], where the nonlinear effects of adaptive algorithm are most significant. The most significant feature of the plot is that the LMS predictors can have smaller MSEs than those of the finite-length Wiener predictor. This is due to the fact that adaptive filters are nonlinear filters. The results from transfer function approach fit well to the simulation results at the region where the nonlinear effects are most significant. The discrepancy of the MSEs predicted by transfer function approach with simulation results at smaller adaptive filter step-size is caused by approximations used in (15), where the frequency of the input signal vector is approximated by that of the center element in that vector.

Fig. 5 plots MSEs of Wiener predictor, standard RLS with chirped AR1 input signal, and rotated RLS with stationary input signal. The signal pole location is at $a = 0.9$, filter length $M = 10$, the prediction distance is 1, chirp rate is $\psi = 5e-4$. It can be seen from the plot that the MSEs of standard RLS on chirped input signal are very close to the MSEs of rotated LMS on stationary input signal.

6. Conclusions

A transform is defined to convert the chirped input signals to baseband stationary input signals, thus provides a different approach in analyzing the tracking performance of the LMS adaptive predictors. This transform makes it possible to apply the error transfer function approach to chirped input signals and compute the total steady-state MSE of the LMS adaptive predictors. It shows that for narrowband input signals, whether stationary or nonstationary, embedded in AWGN, the LMS adaptive predictor may outperform the finite-length Wiener predictor in steady-state MSE. The same chirp transform can be applied to the 1-stp exponentially weighted RLS predictor with chirped input signals.

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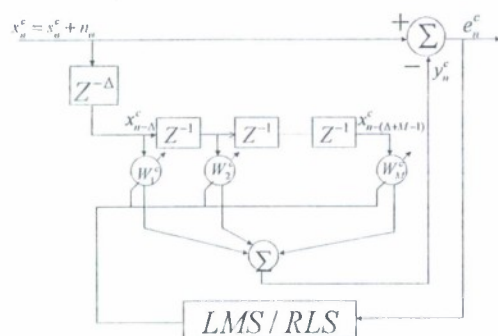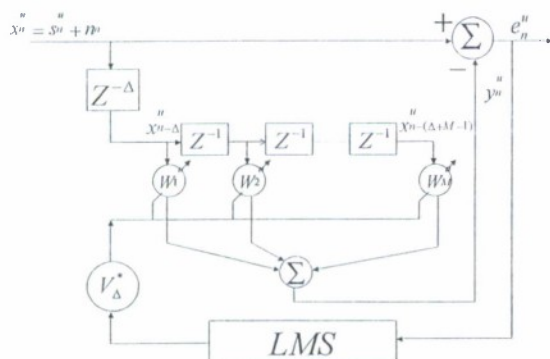
Figure 1. LMS/RLS Δ-step predictor structure.

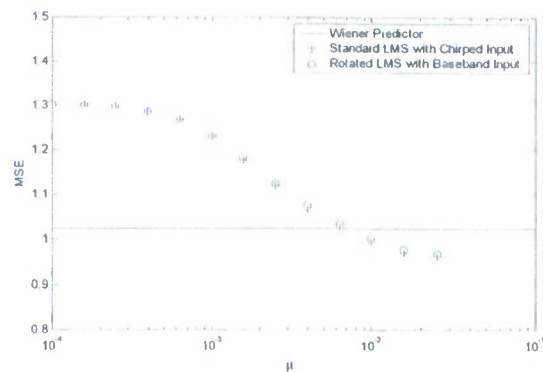Figure 2. Rotated LMS Δ-step predictor structure.

Figure 3. MSEs of Wiener predictor, standard LMS with chirped AR1 input signal, and rotated LMS with stationary input signal. The signal pole location is at $a = 0.9$, filter length $M = 10$, the prediction distance is 5, chirp rate is $\psi = 5e - 4$.
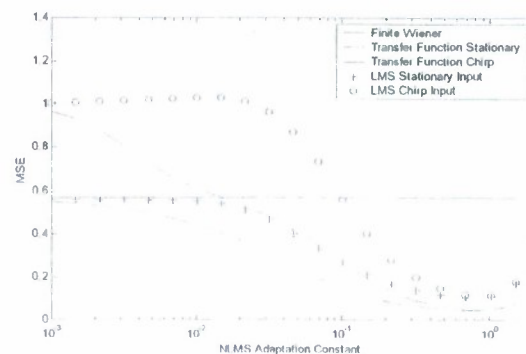

Figure 4. MSEs of 40-step NLMS predictors as a function of adaptation constant, with SNR = 20dB, M=25, a = 0.99, chirp rate $\psi = 5\pi e - 4$.
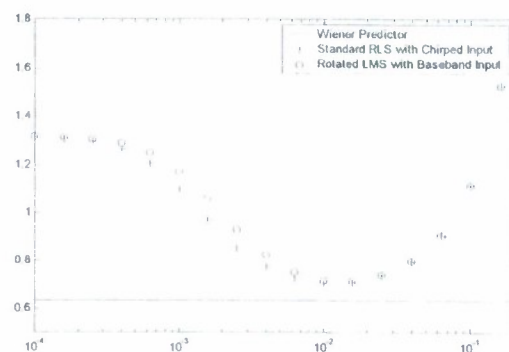

Figure 5. MSEs of Wiener predictor, standard RLS with chirped AR1 input signal, and rotated RLS with stationary input signal. The signal pole location is at $a = 0.9$, filter length $M = 10$, the prediction distance is 1, chirp rate is $\psi = 5e - 4$.